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Corrections to Scaling for Period Doubling

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We have studied a multiple scaling which describes corrections to scaling. For the period doubling in one-dimensional dissipative maps, two-dimensional area-preserving maps, and four-dimensional symplectic maps, the multiple scaling is seen to be well-obeyed, and new scaling factors have been found. The multiple scaling is also seen to be a very powerful tool for searching for scaling behavior.

KEY WORDS: Period doubling; map; critical value; parameter convergence rate; orbital scaling factor; correction to scaling.

1. INTRODUCTION

Feigenbaum discovered[1] universal scaling behavior in the period-doubling route to chaos. The transition to chaos is characterized by two scaling exponents, a parameter convergence rate δ and an orbital scaling factor α. For a set of critical values of the parameter a_n (n denotes the order of period-doubling bifurcation), $\{a_n\} = \{a_n,\ n = 0, 1, 2,...\}$, their difference $\Delta a_n = a_n - a_{n-1}$ converges geometrically to zero, i.e., when n is large,

$$\Delta a_n = C\delta^{-n} \tag{1}$$

where $\delta = 4.6692$ for one-dimensional quadratic maps. For a sequence of orbital elements $\{x_n\}$ one has a similar scaling: when n is large

$$\Delta x_n = C^{(x)}\alpha^{-n} \tag{2}$$

where $\alpha = 2.5029$ for the quadratic maps.

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Previously MacKay[2] found corrections to scaling of the form $\delta^n \Delta a_n \sim C_1 + C_2 \delta'^n$. He interpeted these corrections as resulting from a contracting eigenvector of the renormalization with the eigenvalue δ'. This is analogous to what happens in critical phenomena.

In this brief report, we will expand this correction to scaling into *both* parameter and orbital sequence in *all* one-, two-, and four-dimensional (1d, 2d, and 4d) maps. We will see that a multiple scaling [i.e., more than one term on the right side of Eq. (1) or Eq. (2)] is well obeyed. New scaling factors (characterizing the corrections to scaling) have been found. Corrections to scaling for period doubling in 1d dissipative maps, 2d area-preserving maps, and 4d symplectic maps are discussed, respectively, in Sections 2–4. A summary of the results is given in Section 4.

2. CORRECTIONS TO SCALING IN ONE-DIMENSIONAL DISSIPATIVE MAPS

The idea of corrections to scaling is to search for a "finer" scaling behavior, if any. We try to generalize Eq. (1) [or Eq. (2)] to, when n is large,

$$\Delta a_n = C_1 r_1^{-n} + C_2 r_2^{-n} + \cdots = \sum_{i=1}^{I} C_i r_i^{-n} \tag{3}$$

where n denotes the order of bifurcation, i the level of scaling, r the scaling factor, C a constant, and I the total number of scaling levels. When $I = 1$, Eq. (3) reduces to the regular scaling (one-term scaling) described by Eq. (1).

We will see that Eq. (3) is indeed well obeyed: the r_1 tends to $\delta = 4.669$ for the quadratic 1d maps when $n \to \infty$ (r_i is arranged in an increasing sequence $r_1 < r_2 < r_3 < \cdots$), and all other r_i tend to a constant individually. Thus, Feigenbaum's one-term scaling (1) becomes the lowest order of approximation (i.e., the first term) of Eq. (3). The rest of the terms of Eq. (3) constitute corrections to the Feigenbaum scaling.

To evaluate the scaling factors r_i in the multiple scaling, we first derive equations for them. Equation (3) gives

$$\Delta a_n - t_1 \Delta a_{n+1} + t_2 \Delta a_{n+2} - \cdots + (-1)^I t_I \Delta a_{n+I} = 0 \tag{4}$$

where

$$t_1 = \sum_{i=1}^{I} r_i, \qquad t_2 = \sum_{i<j}^{I} r_i r_j, \qquad t_3 = \sum_{i<j<k}^{I} r_i r_j r_k, \ldots, \qquad t_I = r_1 r_2 \cdots r_I \tag{5}$$

Thus, the scaling factors are solutions of the following eigenvalue equation:

$$r^I - r^{I-1}t_1 + r^{I-2}r_2 - \cdots + (-1)^m r^{I-m} t_m + \cdots + (-1)^I t_I = 0 \qquad (6)$$

To evaluate r_i, one first calculates the t_i by Δa_n using Eq. (4), and then calculates the r_i by t_i using Eq. (6).

We have calculated the scaling factors (r_i) of the parameter sequence $\{a_n\}$ for several 1d maps, and found that the multiple scaling is always very well obeyed. In Table I, results for the map $x' = 1 - ax^2$ are listed. For this quadratic map, the multiple scaling can be expressed as

$$\Delta a_n = \frac{C_1}{\delta^n} + \frac{C_2}{(\delta^2)^n} + \frac{C_3}{(\delta^2\delta')^n} + \frac{C_4}{(\delta^2\delta'')^n} + \frac{C_5}{(\delta^3)^n} + \frac{C_6}{(\delta^3\delta')^n} + \cdots \qquad (7)$$

Table I. Corrections to Scaling for the Parameter of the Map $x' = 1 - ax^2$

	Scaling factors r_i in the multiple scaling, Eq. (3)		
n	$r_1 = \delta$	r_2	r_3
9	4.6692016253334106782	21.7957136	−38.8273
10	4.6692016081444497285	21.8031908	−38.2189
11	4.6692016091403450356	21.8011674	−38.9390
12	4.6692016091001597539	21.8015570	−38.8505
13	4.6692016091031000947	21.8014265	−38.7966
14	4.6692016091029806731	21.8014530	−38.7801
15	4.6692016091029909973	21.8014426	−38.7680
16	4.6692016091029906319	21.8014446	−38.7650
17	4.6692016091029906728	21.8014436	−38.7621
18	4.6692016091029906718	21.8014448	−38.7616

	The Factorization of r_i	
n	r_2/δ	r_3/δ^2
9	4.667944	−1.78095
10	4.6695757	−1.75304
11	4.6691424	−1.74020
12	4.6692258	−1.73614
13	4.6691979	−1.73367
14	4.6692036	−1.73291
15	4.6692013	−1.73236
16	4.6692018	−1.73222
17	4.6692016	−1.73209
18	4.6692016	−1.73207

where

$$\delta = 4.66920160910299067, \qquad \delta' = -1.7320, \qquad \delta'' = -3.739 \tag{8}$$

One can see that the multiple scaling is very powerful for searching for δ: its results to the 11th bifurcation give the same accuracy for δ as that of the one-term scaling to the 20th bifurcation. The δ' and δ'' are *new* scaling factors.

The new scaling factor δ' and δ'' (as well as the Feigenbaum constant δ, of course) are *universal* in a family of maps[3] with $z = 2$, where z is order of the local maximum and thus is the order of the first nonzero term (except constant term) in the corresponding Feigenbaum invariant function

$$g(x) = 1 + \sum_{i=1}^{\infty} a_i x^{zi} \tag{9}$$

However, when z varies, all δ, δ', and δ'' vary. For instance, in a family of maps with $z = 4$,

$$\delta = 7.2846862170733, \qquad \delta' = 3.426, \qquad \delta'' = -4.0 \tag{10}$$

Moreover, the δ' and δ'' (as well as δ) should be, in principle, discernible in the power spectrum, although in practice it is very difficult to have such high resolution.

For "central" (i.e., around the superstable fixed point) orbital sequence $\{x_n\}$ in period doubling, Young[4] has found that the *parameter* scaling factor δ appears in the *orbital* scaling, i.e., $\Delta x_n = C_1 \alpha^{-n} + C_2(\alpha\delta)^{-n}$. When using more terms in the multiple scaling, we found that the new parameter scaling factors δ' and δ'' also appear in the orbital scaling. For the map $x' = 1 - ax^2$, we have

$$\begin{aligned} \Delta x_n = {} & C_1(\alpha)^{-n} + C_2(\alpha\delta)^{-n} + C_3(\alpha^3)^{-n} + C_4(\alpha\delta\delta')^{-n} \\ & + C_5(\alpha\delta\delta'')^{-n} + C_6(\alpha\delta^2)^{-n} + \cdots \end{aligned} \tag{11}$$

where δ, δ', and δ'' are listed in Eq. (8), and

$$\alpha = -2.5029078750958928 \tag{12}$$

It is seen that the multiple scaling in the 12th bifurcation gives the same accuracy of α as that of the one-term scaling in the 20th bifurcation.

3. CORRECTIONS TO SCALING IN TWO-DIMENSIONAL AREA-PRESERVING MAPS

The idea of corrections to scaling for 1d maps described in the previous section can be extended to 2d maps. Consider a 2d area-preserving map, such as the Hénon map

$$x' = \pm y + 1 - ax^2, \qquad y' = \mp x \tag{13}$$

Let a_n denote the critical value of the parameter, and $(x_n^{(0)}, y_n^{(0)})$ the coordinates of the fixed points of a 2^n-cycle on the dominant symmetry line. To study the scaling for parameter and orbital elements along and across the dominant symmetry line, one constructs sequences $\{a_n\}$, $\{x_n^{(0)} - x_n^{(1/2)}\}$, and $\{y_n^{(1/4)}\}$, where $\frac{1}{2}$ and $\frac{1}{4}$ denote the half-way and the quarter-way elements of the cycle, respectively. We found that all three sequences again obey the multiple scaling:

$$\begin{aligned}
\Delta a_n &= C_1\delta^{-n} + C_2(\delta\delta')^{-n} + C_3(\delta^2)^{-n} + \cdots \\
\Delta(x_n^{(0)} - x_n^{(1/2)}) &= C_1^{(x)}\alpha^{-n} + C_2^{(x)}(\alpha^2)^{-n} + C_3^{(x)}(\alpha\delta')^{-n} \\
&\quad + C_4^{(x)}(\alpha\delta)^{-n} + C_5^{(x)}(\alpha^3)^{-n} + \cdots \\
\Delta y_n^{(1/4)} &= C_1^{(y)}\beta^{-n} + C_2^{(y)}(\alpha\beta)^{-n} + C_3^{(y)}(\beta\delta')^{-n} \\
&\quad + C_4^{(y)}(\beta\delta)^{-n} + C_5^{(y)}(\beta^2)^{-n} + \cdots
\end{aligned} \tag{14}$$

$$\begin{aligned}
\delta &= 8.72109720060340, &\quad \delta' &= -8.57384 \\
\alpha &= -4.018076704798909, &\quad \beta &= 16.363896879529
\end{aligned} \tag{15}$$

These δ, α, and β values are the same as, but with much higher accuracy than, those obtained from the one-term scaling. The δ' cannot be factored into δ, α, and β.

4. CORRECTIONS TO SCALING IN FOUR-DIMENSIONAL SYMPLECTIC MAPS

For a 4d Hénon-like symplectic map,[5]

$$\begin{aligned}
x' &= -y + 1 - ax^2 - b(x+z) \\
z' &= -t + 1 - az^2 - b(x-z) \\
y' &= x \\
t' &= z
\end{aligned} \tag{16}$$

a similar analysis of corrections to scaling can be performed. We will therefore only give the results.

The parameter sequences obey the following multiple scaling:

$$\Delta A_n = C_1^{(A)}\delta_1^{-n} + C_2^{(A)}\delta_2^{-n} + \cdots \tag{17}$$

where A could be a or b, $C_i^{(A)}$ (a constant) for a is different from that for b, and

$$\delta_1 = 8.721096, \qquad \delta_2 = -15.0787 \tag{18}$$

For the scalar sequences $\{x_n^{(0)} - x_n^{(1/2)}\}$ and $\{z_n^{(0)} - z_n^{(1/2)}\}$ on the dominant symmetry surface, the multiple scaling is again seen to be very well obeyed,

$$\Delta(X_n^{(0)} - X_n^{(1/2)}) = C_1^{(X)}\alpha_1^{-n} + C_2^{(X)}(\alpha_1^2)^{-n} + C_3^{(X)}(\alpha_1\alpha')^{-n} + C_4^{(X)}(\alpha_1\delta_1)^{-n} + C_5^{(X)}(\alpha_1^2\alpha')^{-n} + \cdots \tag{19}$$

where X could be x or z, $C_i^{(X)}$ (a constant) for x is different from that for z, and

$$\alpha_1 = -4.018076704, \qquad \alpha' = -8.5 \tag{20}$$

For the sequence $\{y_n^{(1/4)}\}$ and $\{t_n^{(1/4)}\}$ across the dominant symmetry surface, the multiple scaling is

$$\Delta Y_n^{(1/4)} = C_1^{(Y)}\beta_2^{-n} + C_2^{(Y)}\beta_1^{-n} + C_3^{(Y)}(\alpha_1\beta_2)^{-n} + C_4^{(Y)}(\alpha_1\beta_1)^{-n} + \cdots \tag{21}$$

where Y could be y or t, $C_i^{(Y)}$ (a constant) for y is different from that for t, and

$$\beta_1 = 16.363, \qquad \beta_2 = -7.53935 \tag{22}$$

5. SUMMARY

We have studied corrections to scaling for both the parameter sequences and the orbital sequences in one-, two-, and four-dimensional maps. We have arrived at the following observations:

1. For a sequence $\{u_n\}$ of critical values in period doubling, u being a parameter or an orbital element, the multiple scaling

$$\Delta u_n = C_1^{(u)}r_1^{-n} + C_2^{(u)}r_2^{-n} + C_3^{(u)}r_3^{-n} + \cdots$$

is very well obeyed when n is large. All the r_i are products of a certain number of "fundamental" scaling factors (a fundamental scaling factor is one that cannot be factored into other scaling factors, e.g., δ, δ', δ'', α, β, etc.; but δ^2, $\alpha\delta$, α^2, etc., are not fundamental).

2. For the parameter sequences, only some of the fundamental scaling factors (i.e., only δ's, not α's or β's) are involved in the corrections to scaling. For orbital sequences, all fundamental scaling factors (α's, β's, and also δ's) are involved.

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